New Variable Separation Excitations of a (2+1)-dimensional Broer-Kaup-Kupershmidt System Obtained by an Extended Mapping Approach

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Using an extended mapping approach, a new type of variable separation excitation with two arbitrary functions of the (2+1)-dimensional Broer-Kaup-Kupershmidt system (BKK) is derived. Based on this excitation, abundant propagating and non-propagating solitons, such as dromions, rings, peakons, compactons, etc. are found by selecting appropriate functions. – PACS: 05.45.Yv, 03.65.Ge

Key words: Extended Mapping Approach; (2+1)-Dimensional BKK System; Localized Excitation.

1. Introduction

In recent studies on soliton theory many powerful approaches were presented. An important approach is multilinear variable separation [1]. In this way, with a Painlevé-Bäcklund transformation, one can find that some significant types of localized excitations, such as dromions, rings, compatons, peakons and loop solutions, can simultaneously exist in a (2+1)-dimensional soliton model [1 – 3]. The main reason is that there is a rather common formula, namely

$$u = \frac{2(a_2a_1 - a_3a_0)q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} \tag{1}$$

with two arbitrary functions $p \equiv p(x,t)$ and $q \equiv q(y,t)$, to describe certain physical quantities of some (2+1)-dimensional models.

Another important approach is the so-called mapping transformation method [4]. The basic idea of this algorithm is that for a given nonlinear partial differential equation (NPDE) with independent variables t, x_1, x_2, $\cdots$, x_m, and dependent variables u,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{2}$$

where P is in general a polynomial function of its arguments, and the subscripts denote partial derivatives, by using the travelling wave transformation, (2) possesses the ansatz

$$u = u(\xi), \quad \xi = \sum_{i=0}^{m} k_i x_i, \tag{3}$$

where k_i with $i = 0, \cdots, m$ are all arbitrary constants. Substituting (3) into (2) yields the ordinary differential equation(ODE) $O(u(\xi), u(\xi)_\xi, u_{\xi\xi}, \cdots) = 0$. Then $u(\xi)$ is expanded into a polynomial in $g(\xi)$

$$u(\xi) = F(g(\xi)) = \sum_{j=0}^{n} a_j g(\xi)^j, \tag{4}$$

where a_j are constants to be determined, and n is fixed by balancing the linear term of the highest order with the nonlinear term in (2). If we suppose $g(\xi) = \tanh(\xi)$, $g(\xi) = \mathrm{sech}(\xi)$ and $g(\xi) = \mathrm{sn}(\xi)$ or $g(\xi) = \mathrm{cn}(\xi)$, respectively, the corresponding approach is usually called the tanh-function method, sech-function method and Jacobian-function method. Although the Jacobian elliptic function method is better known than the tanh-function and sech-function method, the repeated calculations are often tedious. The main idea of the mapping approach is, that $g(\xi)$ is not assumed to be a specific function, such as tanh, sech, sn, cn, etc., but a solution of a mapping equation such as the Riccati equation ($g_\xi = g^2 + a_0$), or a solution of the cubic nonlinear Klein-Gordon equation ($g_\xi^2 = a_4 g^4 c + a_2 g^2 + a_0$), or a solution of the general elliptic equation ($g_\xi^2 = \sum_{i=0}^{4} a_i g^i$), where $a_i, i \in (1,2,3,4)$ are all arbitrary constants. Using the mapping relation (4) and the solutions of these mapping equations, one can obtain many explicit and exact travelling wave solutions of (2).

Now its an interesting question whether all above localized excitations based on the former multilinear approach can be derived by the latter mapping approach, which is usually used to search for travelling wave solutions. The crucial question is how to obtain solutions of (2) with certain arbitrary functions.

In order to derive new solutions with certain arbitrary functions, we assume solutions of the form [5]

$$u = \sum_{i=0}^{n} \alpha_i(x)\phi^i(\omega(x)) \tag{5}$$

with

$$\phi' = \sigma + \phi^2, \tag{6}$$

where $x = (x_0 = t, x_1, x_2, \cdots, x_m)$, σ is a constant and the prime denotes differentiation with respect to ω. To determine u explicitly, one may take the following steps: First, similar to the usual mapping approach, determine n by balancing the highest nonlinear terms and the highest-order partial terms in the given NPDE. Second, substituting (5) and (6) into the given NPDE and collecting coefficients of polynomials of ϕ, then eliminating each coefficient to derive a set of partial differential equations of $\alpha_i(i = 0, 1, ..., n)$and ω. Third, solving the system of partial differential equations to obtain α_i and ω. Finally, as (6) possesses the general solution

$$\phi = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\omega) & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\omega) & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}\omega) & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\omega) & \sigma > 0, \\ -1/\omega & \sigma = 0, \end{cases} \tag{7}$$

substituting α_i, ω and (7) into (5), one can obtain solutions of the concerned in NPDE.

2. New Variable Separation Solutions of the (2+1)-dimensional BKK System

As a concrete example, we consider the celebrated (2+1)-dimensional Broer-Kaup-Kupershmidt(BKK) system [6]

$$\begin{aligned} &H_{yt} - H_{xxy} + 2(HH_x)_y + 2v_{xx} = 0, \\ &v_t + 2(vH)_x + v_{xx} = 0. \end{aligned} \tag{8}$$

The BKK system was used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth. It can also be derived from the celebrated Kadomtsev-Petviashvili(KP) equation by the symmetry constraint [7]. Abundant propagating localized excitations were derived by Lou [1] with help of a Painlevé-Bäcklund transformation and a multilinear variable separation approach. However, as far as we know, its non-propagating solitons has not been reported. It is worth mentioning that this system has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, etc. So, a good understanding of more solutions of the BKK system (8) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model in a harbor and coastal design.

To solve the (2+1)-dimensional Broer-Kaup-Kupershmidt system, we consider the following Painlevé-Bäcklund transformation for H and v in (8):

$$H = (\ln f)_x + H_0, \quad v = (\ln f)_{xy} + v_0, \tag{9}$$

which can be derived from the standard Painlevé truncated expansion, where the functions $H_0 = H_0(x,t)$ and $v_0 = 0$ are seed solutions of the BKK system (8). Based on (9) and the seed solutions, we can straightly obtain a simple relation for H and $v = H_y$. Inserting $v = H_y$ into (8) yields

$$\partial_y(H_t + H_{xx} + 2HH_x) = 0. \tag{10}$$

For simplicity and convenience of discussions, here we take

$$H_t + H_{xx} + 2HH_x = 0. \tag{11}$$

Now we apply the extended mapping approach to (11). By balancing the highest nonlinear term and the highest-order partial term, ansatz (5) becomes

$$H = f(x,y,t) + g(x,y,t)\phi(q(x,y,t)), \tag{12}$$

where $f \equiv f(x,y,t), g \equiv g(x,y,t)$ and $q \equiv q(x,y,t)$ are arbitrary functions of $\{x,y,t\}$ to be determined. Substituting (12) together with (6) into (11), and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, we have

$$6gq_xq_y(q_x + g) = 0, \tag{13}$$

$$\begin{aligned} &2gq_yq_t + 4gq_xg_y + 2g^2q_{xy} + 2gq_yq_{xx} + 2g_yq_x^2 \\ &+ 4gq_xq_{xy} + 4g_xq_yq_x + 4fgq_xq_y + 4g_xgq_y = 0, \end{aligned} \tag{14}$$

$$\begin{aligned}&gq_{xxy}+g_yq_{xx}+2f_xgq_y+2fgq_{xy}+2gq_xf_y\\&+2gg_{xy}+2fg_yq_x+g_yq_t+8g^2q_xq_y\sigma\\&+2g_xq_{xy}+8gq_yq_x^2\sigma+2fg_xq_y+2g_xg_y\\&+2g_{xy}q_x+g_tq_y+g_{xx}q_y+gq_{yt}=0,\end{aligned} \tag{15}$$

$$\begin{aligned}&2g_yq_x^2\sigma+2gq_yq_t\sigma+2fg_{xy}+2g_xf_y+g_{yt}\\&+4g_xq_xq_y\sigma+g_{xxy}+4gq_xg_y\sigma+2f_xg_y\\&+2g^2q_{xy}\sigma+4gq_xq_{xy}\sigma+4g_xgq_y\sigma\\&+4fgq_yq_x\sigma+2gf_{xy}+2gq_yq_{xx}\sigma=0,\end{aligned} \tag{16}$$

$$\begin{aligned}&g_yq_{xx}\sigma+gq_{xxy}\sigma+g_tq_y\sigma+2g^2q_xq_y\sigma^2\\&+f_{xxy}+f_{yt}+gq_{yt}\sigma+2fg_yq_x\sigma+2f_xf_y\\&+2f_xgq_y\sigma+2fg_xq_y\sigma+g_yq_t\sigma+2gq_xf_y\sigma\\&+2fgq_{xy}\sigma+g_{xx}q_y\sigma+2gq_x^2q_y\sigma^2+2g_{yx}q_x\sigma\\&+2g_xq_{xy}\sigma+2ff_{xy}=0.\end{aligned} \tag{17}$$

According to (13) – (17) and through careful calculations, we finally obtain the following exact solution:

$$g=-q_x, \tag{18}$$

$$f=-\frac{q_{xx}+q_t}{2q_x}, \tag{19}$$

$$q=\chi(x,t)+\varphi(y), \tag{20}$$

where $\chi\equiv\chi(x,t), \varphi\equiv\varphi(y)$ are two arbitrary variable separation functions of $\{x,t\}$ and y, respectively.

Now, based on the solutions of (6), one can obtain new types of variable separation excitations of (8).

Case 1. For $\sigma<0$, we can derive the following solitary wave solutions of (8):

$$H_1=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{-\sigma}\chi_x\tanh(\sqrt{-\sigma}(\chi+\varphi)), \tag{21}$$

$$v_1=\sigma\chi_x\varphi_y-\sigma\chi_x\varphi_y\tanh^2(\sqrt{-\sigma}(\chi+\varphi), \tag{22}$$

$$H_2=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{-\sigma}\chi_x\coth(\sqrt{-\sigma}(\chi+\varphi)), \tag{23}$$

$$v_2=\sigma\chi_x\varphi_y-\sigma\chi_x\varphi_y\coth^2(\sqrt{-\sigma}(\chi+\varphi), \tag{24}$$

where $\chi\equiv\chi(x,t)$ and $\varphi\equiv\varphi(y)$ are two arbitrary functions of the indicated variables.

Case 2. For $\sigma>0$, we can obtain the following periodic wave solutions of (8):

$$H_3=-\frac{\chi_{xx}+\chi_t}{2\chi_x}-\sqrt{\sigma}\chi_x\tan(\sqrt{\sigma}(\chi+\varphi)), \tag{25}$$

$$v_3=-\sigma\chi_x\varphi_y-\sigma\chi_x\varphi_y\tan^2(\sqrt{\sigma}(\chi+\varphi)), \tag{26}$$

$$H_4=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{\sigma}\chi_x\cot(\sqrt{\sigma}(\chi+\varphi)), \tag{27}$$

$$v_4=-\sigma\chi_x\varphi_y-\sigma\chi_x\varphi_y\cot^2(\sqrt{-\sigma}(\chi+\varphi)), \tag{28}$$

with two arbitrary functions $\chi(x,t)$and $\varphi(y)$.

Case 3. For $\sigma=0$, we get the following variable separation solution of (8):

$$H_5=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\frac{\chi_x}{\chi+\varphi}, \tag{29}$$

$$v_5=-\frac{\chi_x\varphi_y}{(\chi+\varphi)^2}, \tag{30}$$

also with two arbitrary functions $\chi(x,t)$and $\varphi(y)$.

3. Some Special Solitons in the (2+1)-dimensional BKK System

Because of the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ in the above cases, the physical quantities u and v may possess rich structures. For example, when $\chi(x,t)=f(kx+ct)$ and $\varphi(y)=g(y)$, where f and g are arbitrary functions of the indicated arguments, then all the solutions of the above cases may show abundant localized and propagating excitations. Obviously, one of the simplest travelling wave excitations can be easily obtained when $\chi(x,t)=kx+ct$ and $\varphi(y)=ly$, where k,l and c are arbitrary constants. Still based on the derived solutions, we may also derive rich stationary localized solutions, which are not travelling wave excitations or not propagating waves [8], just as Wu et al. reported about the non-propagating solitons in 1984 [9]. For instance, when the arbitrary functions are selected to be $\chi(x,t)=\varsigma(x)+\tau(t)$ and $\varphi(y)=g(y)$, where ς, τ and g are also arbitrary functions of the indicated arguments, then we can obtain many kinds of non-propagating localized solutions like dromion, ring, peakon, compacton solutions, and so on. Moreover, if $\tau(t)$ is considered to be a periodic function or a solution of a chaos system like the Lorenz chaos system [10], then the non-propagating solitons possess periodic or chaotic behaviors. For simplification of the following discussion, we merely analyse some special localized excitations of solutions v_1 and rewrite (22) in a simple form, namely

$$V\equiv-v_1=-\chi_x\varphi_y\sigma\sec h^2(\sqrt{-\sigma}(\chi+\varphi)),\quad(\sigma<0). \tag{31}$$

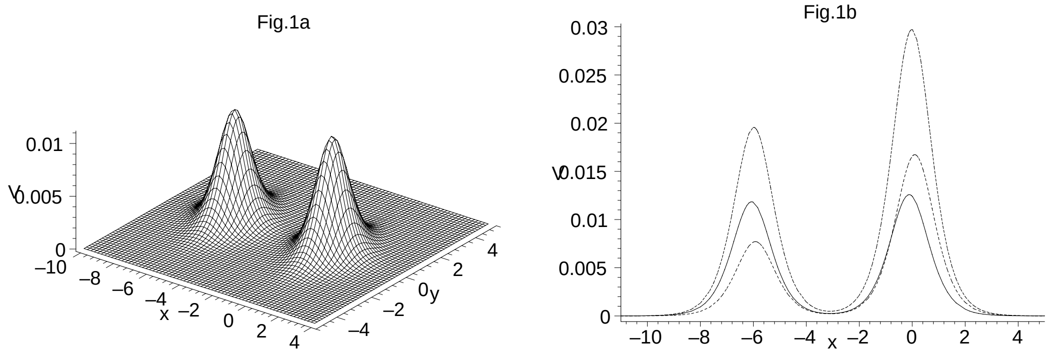

Fig. 1: a) A non-propagating two-dromion structure of the solution v_1 expressed by (31) with the condition (32) at $t = \pi$. b) The corresponding evolutional plot related to a) at $y = 0$ and at the times $t = -2$ (doted line), $t = 0$ (dashed line), $t = 2$ (solid line).

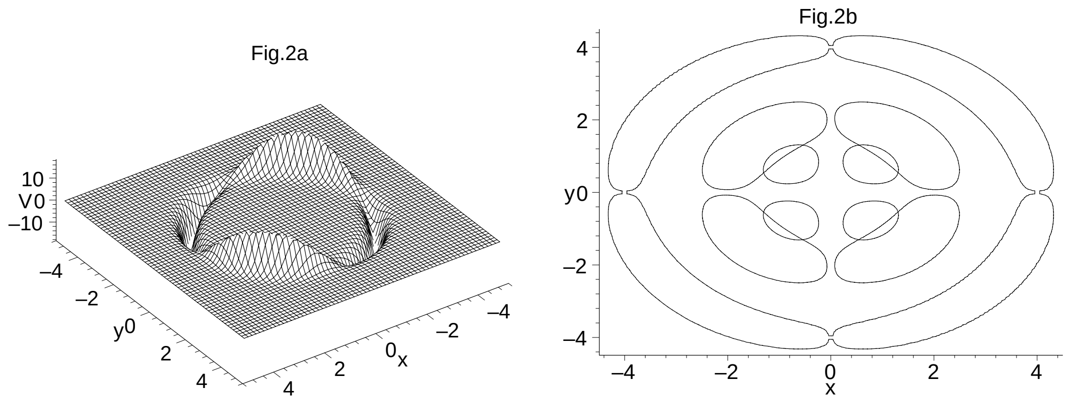

Fig. 2: a) A standing ring breather structure of the solution v_1 expressed by (31) with the condition (34) at time $t = 4$. b) The evolutional contour plot related to a). The value of the contour figure is set $|V| = 0.5$ at the times $t = 0$, $t = 2$ or -2 and $t = 4$ or -4, respectively from inside to outside.

3.1. Non-propagating Solitons

According to the solution v_1 (31), and setting $\sigma = -1$, we first discuss its dromion or dromion lattice excitation, which is one of the significant localized excitations, localized in all directions. When the simple selections of the functions χ and φ are considered to be

$$\chi = 0.1\tanh(x) + 0.1\tanh(x+6) + \sin(t), \quad \varphi = 0.1\tanh(y), \tag{32}$$

then we can obtain a non-propagating two-dromion excitation for the physical quantity v_1 depicted in Figure 1a. The corresponding evolutional plot is presented in Figure 1b. Furthermore, if taking

$$\chi = \sum_{n=-N}^{N} 0.1\tanh(x+6n) + \sin(t),$$

$$\varphi = \sum_{m=-M}^{M} 0.1\tanh(y+6m), \tag{33}$$

then we can obtain a $(2M+1) \times (2N+1)$ "dromion lattice" excitation for the physical field v_1, where M and N are arbitrary positive integers.

In high dimensions, in addition to the point-like localized coherent excitations, there may be some other types of physically significant localized excitations. For instance, in (2+1)-dimensional cases, there are some types of ring soliton solutions which are not identically zero at some closed curves and decay exponentially away from the closed curves. Based on the solution v_1 (31) ($\sigma = -1$), for example, when the functions χ and φ are simply considered to be

$$\chi = -x^2 + t^2, \varphi = -y^2, \tag{34}$$

then we can obtain a ring solition excitation for the physical quantity v_1. Meanwhile, one can find that the

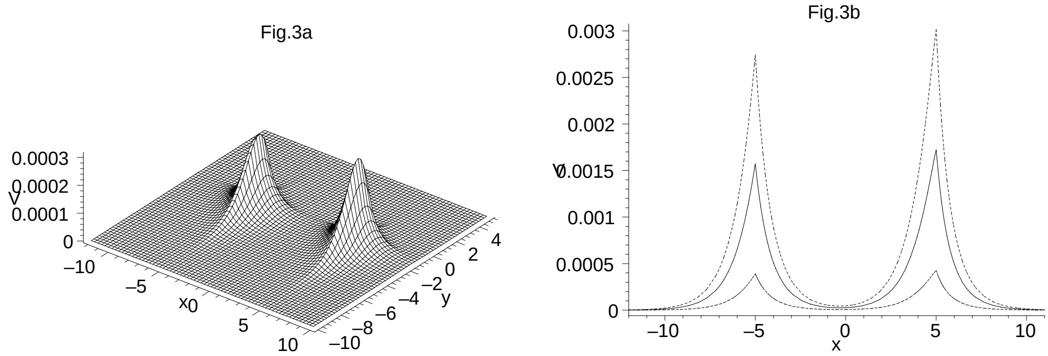

Fig. 3. a) A non-propagating two-peakon soliton plot of the solution v_1 expressed by (31) with the conditions (37) and (38) at $t = 0$. b) The corresponding evolutional plot, related to a) at $y = -2$ and at times $t = -3$ (doted line), $t = 1$ (dashed line) and $t = 3$ (solid line).

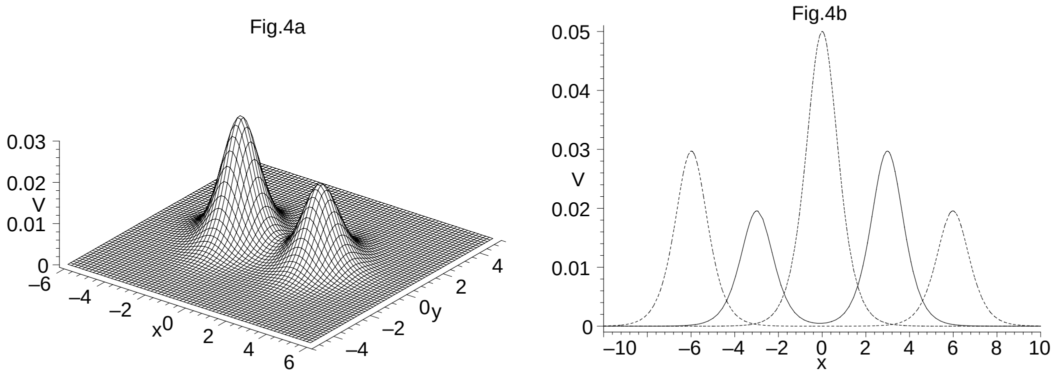

Fig.4. a) A propagating two-dromion structure of the solution v_1 expressed by (31) with condition (39) at $t = -3$. b) The corresponding evolutional plot related to a) at $y = 0$ and at times $t = -6$ (doted line before collision), $t = 0$ (dashed line in collision), $t = 3$ (solid line after collision).

ring soliton solution is a standing breather excitation of the (2+1)-dimensional BKK system. The related evolutional plots are presented in Figure 2.

Still according to the solution v_1 (31), we can obtain some weakly localized excitations such as peakon and compaction excitations [11, 12]. For example, when selecting χ and φ to be some piecewise smooth functions, then we can derive some multipeakon excitations, say

$$\chi = \tau(t) + \sum_{i=1}^{M} \begin{cases} X_i(x), & x \le 0, \\ -X_i(-x) + 2X_i(0), & x > 0, \end{cases} \quad (35)$$

$$\varphi = 1 + \sum_{i=1}^{N} \begin{cases} Y_i(y), & y \le 0, \\ -Y_i(-y) + 2Y_i(0), & y > 0, \end{cases} \quad (36)$$

where the functions $X_i(x)$ and $Y_i(y)$ are differentiable functions of the indicated arguments and possess the boundary conditions $X_i(\pm\infty) = C_{\pm i}, (i = 1, 2, \ldots, M)$ and $Y_i(\pm\infty) = D_{\pm i}, (i = 1, 2, \ldots, N)$, where $C_{\pm i}$ and $D_{\pm i}$ are constants and/or infinity. For instance, when choosing $\tau = \exp(\sin(t))$ and

$$X_1 = 0.1\exp(x+5), X_2 = 0.2\exp(x-5), \quad (37)$$

$$Y_1 = 0.1\exp(y+1), M = 2, N = 1, \quad (38)$$

then we can derive a non-propagating two-peakon excitation for the solution v_1 expressed by (31). The corresponding two-peakon excitation plot is presented in Figure 3. Similarly, if χ and φ are fixed to be other types of piecewise smooth functions similar to [1], then we can derive compacton excitations. Since similar situations have been discussed in [1−3], the related plots are neglected in the present paper.

3.2. Propagating Solitons

Obviously, in the above cases, if the independent variable x is replaced by $x + ct$, since the function

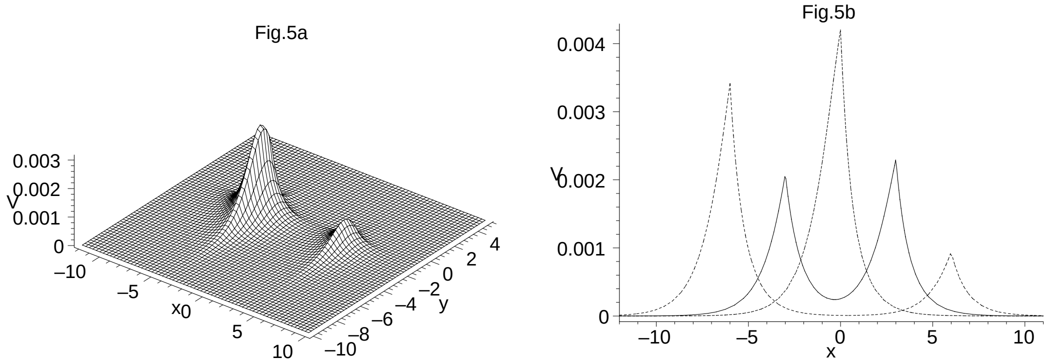

Fig. 5. a) A propagating two-peakon structure of the solution v_1 expressed by (31) with the conditions $X_1 = 0.1\exp(x+t), X_1 = 0.2\exp(x-t), Y_1 = 0.1\exp(y+1)$ in (35) and (36) at $t = -4$. b) The corresponding evolutional plot related to a) at $y = 0$ and at times $t = -6$ (doted line before collision), $t = 0$ (dashed line in collision), $t = 3$ (solid line after collision).

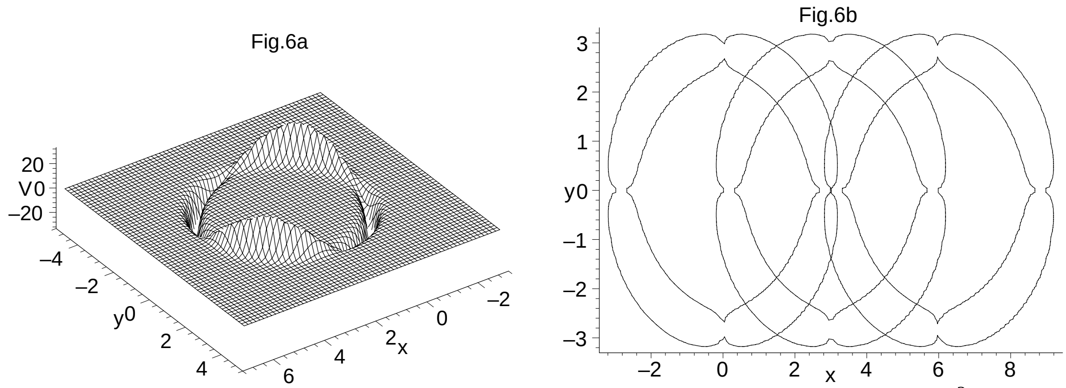

Fig. 6. a) A propagating ring structure of the solution v_1 expressed by (31) with the conditions $\chi = -(x-t)^2+4, \varphi = -y^2+4$ at $t = 2$. b) The corresponding evolutional contour plot related to a). The value of the contour figure is set $|V| = 0.5$ at times $t = 0$, $t = 3$, $t = 6$, travelling from left to right.

$\chi(x,t)$ is an arbitrary function $\{x,t\}$, all the above non-propagating solitons will become propagating solitons. For example, when the functions χ and φ are

$$\chi = 0.1\tanh(x+t) + 0.15\tanh(x-t),\quad \varphi = 0.1\tanh(y), \tag{39}$$

then we can obtain the simple travelling two-dromion excitation for the physical quantity v_1 depicted in Figure 4a. The corresponding evolutional plot is presented in Figure 4b.

Similarly, the propagating peakons and ring solitons can be derived. Two simple examples are presented in Figs. 5 and 6, respectively. The corresponding parameters are presented in the figure legends. It is interesting to mention that by the evolutional Figs. 4b and 5b we have verified the preceding results [1]: for the dromion-dromion collision the interaction is completely elastic while for the peakon-peakon interaction it is nonelastic, since the former preserve their shapes and velocities (right velocity along with positive x axis $c_r = 1$ and left velocity along with negative x axis $c_l = -1$) while the latter change their shapes even though they preserve their velocities after the collision.

4. Summary and Discussion

In summary, by means of an extended mapping approach, the (2+1)-dimensional BKK system is successfully solved. Based on the derived new types of variable separation solutions with two arbitrary functions, we have found rich localized excitations, which are propagating and non-propagating, by selecting the arbitrary functions appropriately. Actually, even according to the solution v_5 (30) are can obtain abundant propagating and non-propagating solitons. The main reason is that, comparing the common formula (1) with the solution (30), one can easily find that they are essentially equivalent when choosing some parameters appropriately. Therefore, all the localized excitations based on the common formula (1) can be re-derived from the solution v_5 (30). This paper is only a beginning work. We expect that the mapping approach applied in our present work may be further extended to other nonlinear physical systems.

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